

Mathematics Embrace the Pilgrim

Imagination & Reasoning in *Purgatory* Canto XVII

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As a pure world of fabrication, imagination overlooks a world of pure reason; it entails the ability to release ourselves from our own narratives, exploring ideas of the mind not present in the senses. Imagination takes visualization one step further to incorporate ideas with no pre-existing mental equipment in the mind.¹ One could point to the world of mathematics — a field whose fundamental nature welcomes abstractness and creativity. In pair with our imaginations, as a theoretical discipline, mathematics is not constrained by the real world. For instance, take axioms of mathematical set theory that can ignite controversy when extending assumptions from the finite to the infinite. Accepting their results can unfold paradoxes that cause one to question if a concept has overstepped the realm of all sense and reasonableness. Yet imperatively, it is crucial that mathematical paradoxes are still consistent with and supported by theorems that are taken to be a necessary truth. Elucidated here is the necessity of imagination to enrich real-world understandings — an idea not unfamiliar to Dante Alighieri, more specifically to *Purgatory*

¹ Refer to Kostopoulos and Cordy. The term “mental equipment” is used in Kostopoulos and Cordy’s exploration of the cognitive space of imagination to mean a “mental picture.”

Canto XVII within *The Divine Comedy*. How is it that fundamentally, a mathematical paradox lends insight to a canto within *The Divine Comedy*? From an exploration of the Banach-Tarski paradox, Dante's "drama of the mind" is illuminated:² the tension between imagining and reasoning. Through this lens, the two powers within the canto are established in balance—that is, fantasy and imagination are the unique central mechanisms for understanding and reasoning.

As one of the strangest results in the history of mathematics, the Banach-Tarski paradox demonstrates the faculty of imagination in disposition with reason. The construction of the paradox relies on a foundational principle known as the Axiom of Choice. In set theory,³ the Axiom of Choice states that, for any collection of sets,⁴ one can construct or choose an element from each set in the collection. Though seemingly direct, the Axiom and its equivalencies play an immense role in contemporary mathematics and are regarded as essential to the study and development of various mathematical branches. When dealing with finite collections of sets, the Axiom's conclusions are indisputable; in fact, "one can even prove its truth" according to Gina Tarrach.⁵ Here lies a particular beauty of mathematical ingenuity, considering mathematicians choose not to be constrained to the finite. Distinctly, two Polish mathematicians, Stefan Banach and Alfred Tarski, uncovered a conclusion unbounded by their

² Refer to Foster. Foster references Francis Fergusson's "Modern Reading of the Purgatorio" in which Fergusson reveals the drama of the order of Dante's vision.

³ The branch of mathematics dealing with properties of well-defined collections of objects.

⁴ Sets in mathematics are organized collections of objects which can range from numbers to functions. For instance, {1,2,3,4} or {blue, white, red} are both sets.

⁵ Refer to Tarrach. Tarrach's exploration on the Axiom of Choice outlines not only the axioms equivalencies, weaker forms, and consequences but also its strength in the discourse of finite sets.

imaginations. That is, one of the Axiom's perplexing consequences in the discourse of infinitely large collections of sets: the Banach-Tarski Paradox.

Introduced by their creators as “a pea chopped up and reassembled into the sun,” the Banach-Tarski paradox states the possibility that a three-dimensional sphere can be disassembled into pieces and reassembled to form two identical copies of the original sphere; no stretching, distorting, or squeezing involved. The results are unsettling, to say the least; how can doubling the volume of an object be possible simply by decomposing and rearranging? In truth, the sphere is not a macro-physical object. It exists in three-dimensional Euclidean space⁶ where the object is represented as an infinite set⁷ of points. The concept of infinity acts not as a measurable quantity but as a conceptualization of the “endless.” When a set of infinite points are constructed (due to the Axiom of Choice) and then divided into measurable subsets, these particular subsets have no defined volume. In turn, partitioning the sphere into cleverly constructed subsets suggests sets that can be turned into one another by rotations and translations. Ultimately, taking an item of infinite points and disassembling it gives each piece the same endless number of points as the original.

⁶ Refer to Tel Aviv. A Euclidean space, often called Cartesian space, is a conception of a two or three-dimensional space that does not connote volume but rather refers to a set of points equipped with some structure. In the case of the Banach Tarski paradox, is it a sphere.

⁷ The infinite set of points involved in Banach-Tarski paradox are said to be countably infinite. Formally, this entails that the members of the set can be put into a one-to-one correspondence with natural numbers (1,2,3,...) considering natural numbers are infinite. In other words, a set of points said to be countably infinite also indicates (like their name suggests) that although its elements are endless, it is still possible to number those elements. Not to mention, the “size” in the set is comprehensible because it is not greater than the “size” of natural numbers. This manipulation of the countably infinite sets is what allows for the conclusions of the Banach-Tarski paradox to be drawn.

Against all intuition, from the concept of infinity, the unsettling statement yields results not too far from the logical.

Though perplexing, through law-abiding intellect, distinguished here is the mind possessed by mathematicians like Stefan Banach and Alfred Tarski. Granting special attention to abstraction and speculation, the faculty of their fantasy and imagination is at play. The power of Banach and Tarski's vision informs their will to saunter into the realm of the infinite and to mark the dawn of our newfound understanding of integral axiom system theory. Imagination and reasoning exhibit their symbiotic relationship: reasoning supports the mathematicians' outcomes and represents the apprehensible value of their claims; imagination, on the other hand, is the moving power for how those claims are reached.

In turn, Dante Alighieri's central exploration of his crafted purgatorial world continues the discourse of imagination and reason, seeing as both principles are dramatically and strategically separated in his seventeenth canto. The text depicts the end of Dante the pilgrims' time in the fourth terrace of Purgatory, reserved for the wrathful. From the first lines of the canto ("Remember, reader" (Purg17.1)), the reader is granted an elaborate visual experience. Exploited is the network of both memory and imagination. Dante writes, "if you have ever been closed in by mountain mist that left you no eyes/ think how those dense damp vapors thinned away slow" (Purg17.3-4). The beginning of the canto focuses on the present visual image: the clearing of the fog, the drawing of the sun to the horizon, and the walk upon a new region of darkness. From a request to the reader's imagination, laid out is a canvas for Dante's visions and reverie, particularly expressed by the switch in attention and by the "O Fantasy"/"O imagination"

(Purg17.13).⁸ In her commentary on the canto XVII, Jo Ann Cavallo brings to light that the place of imagination here “is not simply theoretical musing, but an anticipation of vivid images that pour into the pilgrim’s mind.”⁹ It is these vivid images that will work to extend beyond the text of their references to epitomize Dante’s constructive and active imagination.

What follows the call to fantasy is the “Rein of Wrath”: three visions that enrapt the pilgrim with stories of those consumed by vengeance, including a classical Greek episode in Ovid’s *Metamorphoses*, an Old Testament scene of the book of Esther, and a story from Virgil’s *Aeneid* telling the story of Italian Queen Amata. Notably, the third vision unfolds the special nature of visions in the discourse of imagination. Queen Amata committed suicide after her daughter’s choice of marriage, which she condemns. As noted by Massimo Verdicchio to carry the consistency of the previous examples of wrath, Amata’s suicide should have been the key moment that left an imprint on Dante’s memory.¹⁰ However, Dante’s vision solely highlights Lavinia’s reproach of her mother: “O Queen, why did your anger make you want to be nothing? ... Now you have lost me. I am the one, Mother, who grieves your ruin...” (Purg.17.35-38). This is a moment absent from Virgil’s original story. Thus, here is the product of Dante’s imagination which does not have a source of ordinary sense-perception but rather

⁸ Either “O Fantasy” or “O imagination” depending on the choice of a Dantean scholar’s translation. By the capital “F” in fantasy, the call to imagination of both the reader and author is significant.

⁹ Refer to Cavallo. Written by Jo Ann Cavallo, “Canto XVII: On Revenge” is a chapter within *Lectura Dantis: Purgatorio, A Canto-by-Canto Commentary*. Cavallo’s exploration of the place of revenge in *Purgatory* XVII illustrates the image and definitions of *ira* (Italian for anger) as well as the displacement of blame and fellowship of the angry throughout the canto.

¹⁰ Refer to Verdicchio’s underscore of the relationship between Dante’s arrangement of the poetic imagination and wrath.

works as an instrument of its own. Imagination here points to a new experience instead of illusions preserved in memory. The working of this faculty is extraordinary, and as Cavallo states, “one of divine illumination.”

The interruption of the pilgrim's imaginal reverie brings the reader to the latter half of the canto and to Virgil's extensive explanation of human love. Ultimately, it is a logical construction of the purgatorial structure. Here, an additional overview of the imagination-reason connection can be found. The third and last image of wrath dissolves almost like a dream, causing Dante to awaken from the shock of a great light upon his face. Representing the radiance of the Angel of Gentleness, the great light purges the pilgrim of his wrath and guides him away from the terrace of the wrathful. Interestingly, a visually incapacitated (whether by the lingering darkness of anger or by the abundance of heavenly light) Dante is engrossed in curiosity and confronted by wonder. John Ciardi notes that Dante's phrasing is to be understood as he was “filled with a yearning that allows a man no rest till he have his wish.”¹¹ This yearning resembles the aforementioned “will” of a mathematician to saunter into the realm of the infinite. Similarly, Dante possesses a similar desire to inquire about the unknown. Having completed his ascent to the fourth terrace, the pilgrim makes his inquisitiveness apparent, urging his guide to disclose what lies ahead: “My kind father, tell me what offense is purged on the circle where we are? Let your words not be still...” (Purg.17.82-84). Just as Banach and Tarski satisfy their fantasy and will through logical exploration, Dante satisfies his curiosity in the same manner through Virgil's

¹¹Refer to Ciardi. Ciardi's authoritative translations of Dante's three canticles (*The Inferno*, *The Purgatorio*, and *The Paradiso*) include extensive notes following each canto, clarifying and postulating meanings and stories within the text.

rational exposition. Launched into speech by Dante's inquiry, Virgil outlines the structure of Purgatory through an analysis of love and reason. Ultimately, the pilgrim captivated by imaginational revery at the canto's beginning becomes the listener by its end, having become receptive to the depths of Virgil's insight.

In alliance with Banach and Tarski, imagination functions as the instrument for enriching understanding towards enlightenment. The focus of reason and morality in the last fifty lines of the canto are made possible by the faculty and mobile power of imagination earlier on in the text. The faculty of Dante's imagination speaks to his intellectual prowess and concentration, which are the aptitudes that prove to highlight the pilgrim's growing understanding and awakened mind. Elucidated by Francis Fergusson, from the aspect of imagination, canto XVII is the dawn of Dante's "grasp of the moral-psychological content of his journey in the light of natural reason."¹² In addition, the pilgrim's mental conceptions feature the "moving center of the whole composition," guiding both the pilgrim and the attentive reader.

Fundamentally, the perplexing yet rational nature of the Banach-Tarski paradox incites the spirits of imagination and reasoning in *Purgatory* XVII. Embodying two distinct rival disciplines—one a formal language of numbers and mathematical symbols and one an artistic language of interpretative words and narratives—the paradox and literary canto still extend nuance to one another. The prowess of vision possessed by Banach and Tarski parallels the newfound faculty

¹² Refer to Fergusson's accentuation of the intellectual and spiritual development of Dante the pilgrim. Fergusson emphasizes the distinctions between Dante the Author and Dante the Pilgrim and the importance of the Pilgrim's growing understanding for the attentive reader.

of pilgrim's mind. As a result, both their faculties compel them towards enriched understanding and contribution to the outside world. Foster's lasting note perfectly encompasses the disjointed but rather beautiful nature of canto XVII: "so much of the human spirit has been shown us, successively, in action: the wonder of the human imagination [complements] the lucid gravity of reasoning."¹³

¹³ Refer to Foster's analysis of the human spirit in action throughout canto XVII, that is, "the wonder of human imagination; memory and visual perception; the lucid gravity of reasoning."

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